



**Singular limit for quasi-linear
diffusive transport through a thin
heterogeneous layer**

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Abstract

Reactive transport processes in porous media including thin heterogeneous layers play an important role in many applications. In this paper, we investigate a quasi-linear reaction-diffusion problem in a domain consisting of two bulk-domains which are separated by a thin layer with a periodic heterogeneous structure. The thickness of the layer, as well as the periodicity within the layer are of order ϵ , where ϵ is much smaller than the size of the bulk-domains. For the singular limit $\epsilon \rightarrow 0$, when the thin layer reduces to an interface, we rigorously derive a macroscopic model with effective interface conditions between the two bulk-domains. Due to the oscillations within the layer, we have to combine dimension reduction techniques with methods from the homogenization theory. To cope with these difficulties, we make use of the two-scale convergence in thin heterogeneous layers. However, in our case the diffusion in the thin layer is low and depends nonlinear on the concentration itself. The low diffusion leads to a two-scale limit depending on a macroscopic and a microscopic variable. Hence, weak compactness results based on standard a priori estimates are not enough to pass to the limit $\epsilon \rightarrow 0$ in the nonlinear terms. Therefore, we derive strong two-scale compactness results based on a variational principle. Further, we establish uniqueness for the microscopic and the macroscopic model.

1 Introduction

Reactive transport processes through thin highly heterogeneous layers occur in a variety of applications, especially from biosciences, medical sciences, material sciences, and geosciences. Important examples are endothelial layers as the endothelium of a blood vessel, which forms the innermost layer of the vessel and acts as a membrane between the vessel wall and the lumen of the vessel. Membranes are thin selective barriers separating bulk-domains, where the physical properties of the membrane differ from the surrounding media, and in many applications, as in the case of the endothelium, membranes carry a highly heterogeneous structure.

Microscopic models for physical processes in such type of media describe the processes taking place on the microscopic scale, as well as the whole complexity of the geometry. Hence, the microscopic problem includes different scales, as the thickness of the layer, the size of the bulk-domains, and the size of the heterogeneity within layer. This leads to high numerical challenges. A possibility to overcome this problem is the derivation of macroscopic models in the singular limit when the thin layer is reduced to an interface separating the bulk-domains. The solution of this macroscopic problem is an approximation of the microscopic model. The challenging point is to find so called effective interface condition across the interface between the bulk-domains. These interface conditions carry information about the microscopic processes within the thin heterogeneous layer.

In this paper we rigorously derive a macroscopic model for a reaction-diffusion problem in a domain Ω_ϵ consisting of the bulk-domains Ω_ϵ^+ and Ω_ϵ^- , which are separated by a thin layer

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Ω_ϵ^M with a periodic heterogeneous structure. The thickness of the layer and the periodicity within the layer are of order ϵ , where the parameter ϵ is small compared to the size of the bulk-domains. Across the interfaces between the bulk-domains and the thin layer we assume continuity of the solution and the normal-fluxes. The diffusion coefficient in the thin layer is oscillating, describing the heterogeneous structure within the layer, and depends nonlinearly on the solution itself. Additionally, we consider a critical scaling of the equation in the thin layer. In the singular limit for $\epsilon \rightarrow 0$ the thin layer Ω_ϵ^M reduces to an interface Σ between two bulk-domains Ω^+ and Ω^- . We show, that the solution u_ϵ of the microscopic model converges to a limit function u_0 , where u_0 is the unique solution of a macroscopic problem with effective interface conditions across Σ .

To pass to the limit $\epsilon \rightarrow 0$ in the thin layer we have to cope simultaneously with the singular limit and the oscillations in the layer. For this we make use of the method of two-scale convergence in thin heterogeneous layers. This method was introduced for thin homogeneous structures in [1] and later generalized to thin periodic structures in [2]. Additionally we have to handle the coupling between the bulk-domains and the thin layer. Due to the nonlinear diffusion in the layer, we need strong two-scale convergence results to pass to the limit.

Due to the specific scaling in the diffusion-equation in the thin layer, the two-scale limit depends on the macroscopic and the microscopic variable. In other words, in the formal two-scale asymptotic expansion the zeroth order term is depending on the macroscopic variable $\bar{x} \in \Sigma$, and the oscillating microscopic variable $y = \frac{x}{\epsilon}$. Additionally, we have to cope with low regularity for the microscopic solution. Therefore, standard averaging methods in the thin layer together with the Aubin-Lions compactness theorem, see e.g., [3], fail to prove the strong convergence. Other methods like the Kolmogorov-compactness theorem used in [3] and [2] for a semi-linear problem fail because of the nonlinear diffusion term. In this paper, we combine methods used in [4] and [5] to establish the strong two-scale convergence in the thin layer. More precisely, in a first step we obtain weak two-scale compactness results in the thin layer and strong compactness in L^2 for the microscopic solutions in the bulk-domains. These convergence results are based on uniform a priori estimates, and the Aubin-Lions Lemma in the bulk-domains. With the help of the Kirchhoff-transformation we are able to pass to the limit $\epsilon \rightarrow 0$ and to derive a variational equation on the macroscopic scale, where this equation includes limit functions for the microscopic solutions and their Kirchhoff-transformations. In the second step, we have to identify these limits in variational equation. This is done by a variational method, where we choose specific test-functions in the microscopic problem, which solve an auxiliary problem. Then we can pass to the limit by controlling the solutions in the thin layer using the continuity across the bulk-layer-interfaces with the solutions in the bulk-domains, for which we already established strong convergence.

Starting from the pioneering work of Sanchez-Palencia [6], there is a rich literature on problems in thin domains with applications in solid mechanics, wave diffraction, porous media and so on. In particular, we have to mention the monography [7], where a variety of different scalings for linear problems are treated by the Bakhvalov-ansatz. Based on this ansatz, error estimates depending on ϵ are derived. However, in this method the a priori knowledge of the macroscopic model and associated cell problems are necessary, and higher regularity results for these problems are important. In contrast, in the present paper we have to deal with low regularity and nonlinear problems. Reaction-diffusion processes through thin heterogeneous layers including nonlinear reaction terms have been considered for different scalings in the thin layer in [8, 3, 2]. Unsaturated Darcy-flow through a thin homogeneous layer is treated in [9] for different scalings. However, the critical scaling when the two-scale limit depends on the macroscopic and microscopic variable is not considered. A linear single phase flow through a thin filter is considered in [10], where the thickness and the periodicity of the filter, as well as the thickness of channels of the filter are of different orders. This leads to different scalings in the equation in the filter and therefore different macroscopic equations are obtained. In [11] a non-Newtonian flow through a thin filter is treated, where an approximation of the microscopic solution is constructed via a macroscopic problem and boundary layers. Therefore, additional regularity for the microscopic flow is

needed. In [12] the same authors considered a linear reaction-diffusion-advection problem through a filter with small obstacles, and constructed an approximation for the microscopic solution by using correctors including solutions of boundary layer problems. In [5] a two phase flow is considered in a perforated domain, but the scaling is closely related to our problem and we use similar methods to prove strong convergence.

The paper is organized as follows: In Section 2 we describe the geometrical setting and introduce the microscopic model with the associated assumptions. Further, we give some basic results about existence, uniqueness, and a priori estimates. In Section 3 we present the main result of this paper. Weak convergence results in the layer and the strong convergence results in the bulk-domains are proved in Section 4. In Section 5 we prove the strong two-scale convergence in the thin layer. In Section 6, we show uniqueness for the microscopic model.

1.1 Novelty and importance of the main result

While many papers deal with linear reactive transport through thin heterogeneous layers, rigorous results about quasi-linear problems with nonlinear diffusion seem to be rare. However, such nonlinear problems are of particular importance in applications, where the properties of the porous media are influenced by the solution itself. To the best of the authors knowledge, for the critical scaling in the thin layer considered in this paper, when the two-scale limit is depending on both, the macroscopic and the microscopic variable, there are no rigorous homogenization results for nonlinear diffusion. Further, the results are based on low regularity results for the microscopic and the macroscopic solution. To pass to the limit in the variational equation including the Kirchhoff-transformation, we introduce an appropriate space of test-functions. For the proof of the strong two-scale convergence in the thin layer we make use of a variational method. For this method, an a priori knowledge of the macroscopic model is not necessary. Further, it is applicable to other types of problems and the application to unsaturated flow is part of an ongoing work. Additionally, we prove uniqueness for the microscopic and the macroscopic model by using a weak entropy condition.

2 The microscopic model

We consider the domain $\Omega_\epsilon := \Sigma \times (-\epsilon - H, H + \epsilon) \subset \mathbb{R}^n$ with fixed $H \in \mathbb{N}$, $n \geq 2$, and $\Sigma = (0, l_1) \times \dots \times (0, l_{n-1}) \subset \mathbb{R}^{n-1}$ with $l = (l_1, \dots, l_{n-1}) \in \mathbb{N}^{n-1}$. Further, let $\epsilon > 0$ be a sequence with $\epsilon^{-1} \in \mathbb{N}$. The set Ω_ϵ consists of three subdomains, see Figure 1, given by

$$\begin{aligned}\Omega_\epsilon^+ &:= \Sigma \times (\epsilon, H + \epsilon), \\ \Omega_\epsilon^M &:= \Sigma \times (-\epsilon, \epsilon), \\ \Omega_\epsilon^- &:= \Sigma \times (-\epsilon - H, -\epsilon).\end{aligned}$$

The domains Ω_ϵ^\pm and Ω_ϵ^M are separated by an interface S_ϵ^\pm , i. e.,

$$S_\epsilon^+ := \Sigma \times \{\epsilon\} \quad \text{and} \quad S_\epsilon^- := \Sigma \times \{-\epsilon\},$$

hence, we have $\Omega_\epsilon = \Omega_\epsilon^+ \cup \Omega_\epsilon^- \cup \Omega_\epsilon^M \cup S_\epsilon^+ \cup S_\epsilon^-$.

As mentioned above, for $\epsilon \rightarrow 0$ the membrane Ω_ϵ^M reduces to an interface $\Sigma \times \{0\}$, which we also denote by Σ suppressing the n -th component, and we define

$$\Omega^+ := \Sigma \times (0, H) \quad \text{and} \quad \Omega^- := \Sigma \times (-H, 0),$$

and $\Omega := \Omega^+ \cup \Sigma \cup \Omega^- = \Sigma \times (-H, H)$. The microscopic structure within the thin layer Ω_ϵ^M can be described by shifted and scaled reference elements. We define

$$\begin{aligned}Y &:= (0, 1)^{n-1}, \\ Z &:= Y \times (-1, 1).\end{aligned}$$

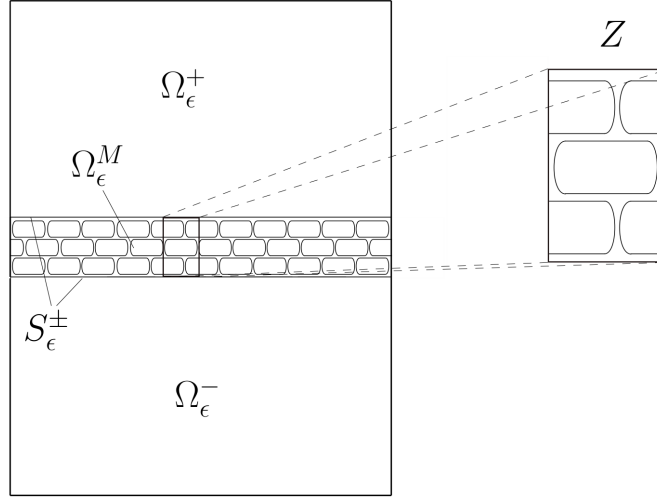


Figure 1: The microscopic domain containing the thin layer Ω_ϵ with periodic structure for $n = 2$. The heterogeneous structure for the thin layer is modeled by the oscillating diffusion coefficient D .

We denote the upper and lower boundary of Z by

$$S^+ := Y \times \{1\} \quad \text{and} \quad S^- := Y \times \{-1\}.$$

Now, we are looking for the microscopic solution $u_\epsilon = (u_\epsilon^+, u_\epsilon^M, u_\epsilon^-)$, with $u_\epsilon^\pm : (0, T) \times \Omega_\epsilon^\pm \rightarrow \mathbb{R}$ and $u_\epsilon^M : (0, T) \times \Omega_\epsilon^M \rightarrow \mathbb{R}$, of the following problem:

$$\begin{aligned}
\partial_t u_\epsilon^\pm - \Delta u_\epsilon^\pm &= f^\pm(u_\epsilon^\pm) && \text{in } (0, T) \times \Omega_\epsilon^\pm, \\
\frac{1}{\epsilon} \partial_t u_\epsilon^M - \epsilon \nabla \cdot \left(a(u_\epsilon^M) D \left(\frac{x}{\epsilon} \right) \nabla u_\epsilon^M \right) &= \frac{1}{\epsilon} g_\epsilon^M && \text{in } (0, T) \times \Omega_\epsilon^M, \\
u_\epsilon^M &= u_\epsilon^\pm && \text{on } (0, T) \times S_\epsilon^\pm, \\
-\epsilon a(u_\epsilon^M) D \left(\frac{x}{\epsilon} \right) \nabla u_\epsilon^M \cdot \nu &= -\nabla u_\epsilon^\pm \cdot \nu && \text{on } (0, T) \times S_\epsilon^\pm, \\
u_\epsilon^M(0) &= u_{\epsilon, i}^M && \text{in } \Omega_\epsilon^M, \\
u_\epsilon^\pm(0) &= u_{\epsilon, i}^\pm && \text{in } \Omega_\epsilon^\pm, \\
-\epsilon a(u_\epsilon^M) D \left(\frac{x}{\epsilon} \right) \nabla u_\epsilon^M \cdot \nu &= 0 && \text{on } (0, T) \times \partial \Omega_\epsilon^M \setminus (S_\epsilon^+ \cup S_\epsilon^-), \\
-\nabla u_\epsilon^\pm \cdot \nu &= 0 && \text{on } (0, T) \times \partial \Omega_\epsilon^\pm \setminus S_\epsilon^\pm.
\end{aligned} \tag{1}$$

The weak formulation of Problem (1) reads as follows: Find $u_\epsilon \in L^2((0, T), H^1(\Omega_\epsilon))$, such that $\partial_t u_\epsilon^\pm \in L^2((0, T), H^{-1}(\Omega_\epsilon^\pm))$ and $\partial_t u_\epsilon^M \in L^2((0, T), H^{-1}(\Omega_\epsilon^M))$, and for all $\phi_\epsilon = (\phi_\epsilon^+, \phi_\epsilon^M, \phi_\epsilon^-) \in C_0^\infty([0, T], H^1(\Omega_\epsilon))$ it holds that

$$\begin{aligned}
& \sum_{\pm} \left[- \int_0^T \int_{\Omega_\epsilon^\pm} u_\epsilon^\pm \partial_t \phi_\epsilon^\pm dx dt + \int_0^T \int_{\Omega_\epsilon^\pm} \nabla u_\epsilon^\pm \cdot \nabla \phi_\epsilon^\pm dx dt \right] \\
& - \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} u_\epsilon^M \partial_t \phi_\epsilon^M dx dt + \epsilon \int_0^T \int_{\Omega_\epsilon^M} a(u_\epsilon^M) D \left(\frac{x}{\epsilon} \right) \nabla u_\epsilon^M \cdot \nabla \phi_\epsilon^M dx dt \\
& = \sum_{\pm} \left[\int_0^T \int_{\Omega_\epsilon^\pm} f^\pm(u_\epsilon^\pm) \phi_\epsilon^\pm dx dt + \int_{\Omega_\epsilon^\pm} u_{\epsilon, i}^\pm \phi_\epsilon^\pm(0) dx \right] \\
& + \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} g_\epsilon^M \phi_\epsilon^M dx dt + \frac{1}{\epsilon} \int_{\Omega_\epsilon^M} u_{\epsilon, i}^M \phi_\epsilon^M(0) dx.
\end{aligned}$$

In the following, we use the subscribe $\#$ to indicate function spaces defined on the domain Z consisting of Y -periodic functions, for example $H^1_{\#}(Z)$ denotes the space of functions in $H^1(Z)$ which are Y -periodic with respect to the first $(n-1)$ th components $\bar{y} := (y_1, \dots, y_{n-1}) \in Y$. Further, for $x \in \mathbb{R}^n$ we write $\bar{x} := (x_1, \dots, x_{n-1})$.

Transformation of the bulk-domains:

For an easier notation we shift the whole bulk-domains Ω_ϵ^\pm to the fixed domains Ω^\pm . However, we keep the same notation as above and consider the following problem: Let us define the space of solutions

$$V_\epsilon := \left\{ u_\epsilon = (u_\epsilon^+, u_\epsilon^M, u_\epsilon^-) \in H^1(\Omega^+) \times H^1(\Omega_\epsilon^M) \times H^1(\Omega^-) : u_\epsilon^\pm|_\Sigma = u_\epsilon^M|_{S_\epsilon^\pm} \right\}.$$

Further, we define the Kirchhoff-transformation

$$\mathcal{A}(s) := \int_0^s a(\eta) d\eta \quad \text{for all } s \in \mathbb{R}.$$

Now, we say $u_\epsilon \in L^2((0, T), V_\epsilon)$ is a weak solution of the Problem (1) iff for all $\phi_\epsilon \in C_0^\infty([0, T], V_\epsilon)$ it holds:

$$\begin{aligned} & \sum_{\pm} \left[- \int_0^T \int_{\Omega^\pm} u_\epsilon^\pm \partial_t \phi_\epsilon^\pm dx dt + \int_0^T \int_{\Omega^\pm} \nabla u_\epsilon^\pm \cdot \nabla \phi_\epsilon^\pm dx dt \right] \\ & \quad - \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} u_\epsilon^M \partial_t \phi_\epsilon^M dx dt + \epsilon \int_0^T \int_{\Omega_\epsilon^M} D\left(\frac{x}{\epsilon}\right) \nabla \mathcal{A}(u_\epsilon^M) \cdot \nabla \phi_\epsilon^M dx dt \\ & = \sum_{\pm} \left[\int_0^T \int_{\Omega^\pm} f^\pm(u_\epsilon^\pm) \phi_\epsilon^\pm dx dt + \int_{\Omega^\pm} u_{\epsilon, i}^\pm \phi_\epsilon^\pm(0) dx \right] \\ & \quad + \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} g_\epsilon^M \phi_\epsilon^M dx dt + \frac{1}{\epsilon} \int_{\Omega_\epsilon^M} u_{\epsilon, i}^M \phi_\epsilon^M(0) dx. \end{aligned} \tag{2}$$

Assumptions on the data:

(A1) $f^\pm : [0, T] \times \overline{\Omega^\pm} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $z \mapsto f(t, x, z)$ is Lipschitz continuous uniformly with respect to (t, x) .

(A2) $g_\epsilon^M \in L^2((0, T) \times \Omega_\epsilon^M)$ and there exists $g_0^M \in L^2((0, T) \times \Sigma \times Z)$, such that

$$g_\epsilon^M \rightarrow g_0^M \quad \text{strongly in the two-scale sense.}$$

(A3) $D \in L^\infty_{\#}(Z)^{n \times n}$ is symmetric and coercive, i. e., there exists $d_0 > 0$ such that

$$D(y)\xi \cdot \xi \geq d_0 \|\xi\|^2 \quad \text{for all } y \in Z, \xi \in \mathbb{R}^n.$$

(A4) $a : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz-continuous and there exist $a_0 > 0$ and $A_0 > 0$, such that

$$0 < a_0 \leq a(s) \leq A_0 \quad \text{for all } s \in \mathbb{R}.$$

(A5) $u_{\epsilon, i}^\pm \in L^2(\Omega^\pm)$ and $u_{\epsilon, i}^M \in L^2(\Omega_\epsilon^M)$ and there exist $u_i^\pm \in L^2(\Omega^\pm)$ and $u_i^M \in L^2(\Sigma \times Z)$ such that

$$\begin{aligned} u_{\epsilon, i}^\pm & \rightarrow u_i^\pm & & \text{in } L^2(\Omega^\pm), \\ u_{\epsilon, i}^M & \rightarrow u_i^M & & \text{strongly in the two-scale sense.} \end{aligned}$$

Due to the nonlinear diffusion in the thin layer, we only obtain the time-derivative of the microscopic solution in the distributional sense in each subdomain Ω^\pm and Ω_ϵ^M . Further, due to the coupling condition, we are not able to show $\partial_t u_\epsilon^M \in L^2((0, T), H^1(\Omega_\epsilon^M)')$ and

$\partial_t u_\epsilon^\pm \in L^2((0, T), H^1(\Omega^\pm)')$. We have to consider the dual space of Sobolev functions with vanishing traces on S_ϵ^\pm . We define:

$$\begin{aligned}\mathcal{H}_\Sigma^1(\Omega^\pm) &:= \{u^\pm \in H^1(\Omega^\pm) : u|_\Sigma = 0\}, \\ \mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M) &:= \{u_\epsilon^M \in H^1(\Omega_\epsilon^M) : u|_{S_\epsilon^\pm} = 0\}, \\ \mathcal{H}_{S^\pm}^1(Z) &:= \{u_0^M \in H^1(Z) : u|_{S^\pm} = 0\}.\end{aligned}$$

Proposition 1. *There exists a unique weak solution u_ϵ of the Problem (1), which fulfills the following a priori estimates*

$$\begin{aligned}\|\partial_t u_\epsilon^\pm\|_{L^2((0, T), \mathcal{H}_\Sigma^1(\Omega^\pm)')} + \|u_\epsilon^\pm\|_{L^\infty((0, T), L^2(\Omega^\pm))} + \|\nabla u_\epsilon^\pm\|_{L^2((0, T) \times \Omega^\pm)} \leq C, \\ \frac{1}{\sqrt{\epsilon}} \|\partial_t u_\epsilon^M\|_{L^2((0, T), \mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M)')} + \frac{1}{\sqrt{\epsilon}} \|u_\epsilon^M\|_{L^\infty((0, T), L^2(\Omega_\epsilon^M))} + \sqrt{\epsilon} \|\nabla u_\epsilon^M\|_{L^2((0, T) \times \Omega_\epsilon^M)} \leq C.\end{aligned}$$

Proof. The existence of a weak solution can be established by the Galerkin method and the homotopy principle of Leray-Schauder, see e. g., [13, Chapter 9.2, Theorem 4]. For the uniqueness see Theorem 2 in Section 6. \square

As an easy consequence of Proposition 1 we obtain the following a priori estimates for the Kirchhoff-transformation $\mathcal{A}(u_\epsilon^M)$.

Corollary 1. *For the solution u_ϵ of Problem (1) it holds that*

$$\frac{1}{\sqrt{\epsilon}} \|\mathcal{A}(u_\epsilon^M)\|_{L^\infty((0, T), L^2(\Omega_\epsilon^M))} + \sqrt{\epsilon} \|\nabla \mathcal{A}(u_\epsilon^M)\|_{L^2((0, T) \times \Omega_\epsilon^M)} \leq C.$$

Remark 1. *We emphasize, that for $\phi_\epsilon^M \in C^\infty([0, t], \mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M))$ with $t \in [0, T]$ it holds that*

$$\begin{aligned}\int_0^t \langle \partial_t u_\epsilon^M, \phi_\epsilon^M \rangle_{\mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M)', \mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M)} dt &= -\frac{1}{\epsilon} \int_0^t \int_{\Omega_\epsilon^M} u_\epsilon^M \partial_t \phi_\epsilon^M dx dt \\ &\quad - \frac{1}{\epsilon} \int_{\Omega_\epsilon^M} u_{\epsilon, i}^M \phi_\epsilon^M(0) dx + \frac{1}{\epsilon} \int_{\Omega_\epsilon^M} u_\epsilon^M(t) \phi_\epsilon^M(t) dx.\end{aligned}$$

3 Main result

In this section, we state the main result of our paper. For the definition of the space V_0 see Section 4.

Theorem 1. *For the sequence of solutions u_ϵ of the Problem (1) it holds for $\beta \in (\frac{1}{2}, 1)$*

$$\begin{aligned}u_\epsilon^\pm &\rightarrow u_0^\pm && \text{in } L^2((0, T), H^\beta(\Omega^\pm)), \\ \nabla u_\epsilon^\pm &\rightarrow \nabla u_0^\pm && \text{weakly in } L^2((0, T) \times \Omega^\pm), \\ u_\epsilon^M &\rightarrow u_0^M && \text{strongly in the two-scale sense,} \\ \epsilon \nabla u_\epsilon^M &\rightarrow \nabla_y u_0^M && \text{in the two-scale sense,}\end{aligned}$$

where $u_0 = (u_0^+, u_0^M, u_0^-) \in L^2((0, T), V_0)$ is the unique weak solution of

$$\begin{aligned}
\partial_t u_0^\pm - \Delta u_0^\pm &= f^\pm(u_0^\pm) && \text{in } (0, T) \times \Omega^\pm, \\
\partial_t u_0^M - \nabla_y \cdot (a(u_0^M) D(y) \nabla_y u_0^M) &= g_0^M && \text{in } (0, T) \times \Sigma \times Z, \\
-\nabla u_0^\pm \cdot \nu &= 0 && \text{on } (0, T) \times \partial\Omega^\pm \setminus \Sigma, \\
u_0^\pm &= u_0^M && \text{on } (0, T) \times \Sigma \times S^\pm, \\
-\nabla u_0^\pm \cdot \nu &= \int_{S^\pm} a(u_0^M) D(y) \nabla_y u_0^M \cdot \nu_{\partial Z} d\sigma_y && \text{on } (0, T) \times \Sigma, \\
u_0^\pm(0) &= u_i^\pm && \text{in } \Omega^\pm, \\
u_0^M(0) &= u_i^M && \text{in } \Sigma \times Z, \\
u_0^M &\text{ is } Y\text{-periodic,} &&
\end{aligned} \tag{3}$$

where $\nu_{\partial Z}$ denotes the unit outer normal on ∂Z with respect to Z .

We call $u_0 = (u_0^+, u_0^M, u_0^-) \in L^2((0, T), V_0)$ a weak solution of the macroscopic problem (3) if for all $\phi = (\phi^+, \phi^M, \phi^-) \in C_0^\infty([0, T], V_0)$ it holds that

$$\begin{aligned}
& \sum_{\pm} \left[\int_0^T \int_{\Omega^\pm} \nabla u_0^\pm \cdot \nabla \phi^\pm dx dt - \int_0^T \int_{\Omega^\pm} u_0^\pm \partial_t \phi^\pm dx dt \right] \\
& - \int_0^T \int_{\Sigma} \int_Z u_0^M \partial_t \phi^M dy d\bar{x} dt + \int_0^T \int_{\Sigma} \int_Z a(u_0^M) D(y) \nabla_y u_0^M \cdot \nabla_y \phi^M dy d\bar{x} dt \\
& = \sum_{\pm} \left[\int_0^T \int_{\Omega^\pm} f^\pm(u_0^\pm) \phi^\pm dx dt + \int_{\Omega^\pm} u_i^\pm \phi^\pm(0) dx \right] \\
& + \int_0^T \int_{\Sigma} \int_Z g_0^M \phi^M dy d\bar{x} dt + \int_{\Sigma} \int_Z u_i^M \phi^M(0) dy d\bar{x}.
\end{aligned}$$

In a first step, see Section 4, we prove the convergence results for u_ϵ^\pm and the (weak) two-scale convergence results for u_ϵ^M and $\mathcal{A}(u_\epsilon^M)$, which hold up to a subsequence. However, this is not enough to pass to the limit in the nonlinear term $a(u_\epsilon^M)$ in the thin layer. Nevertheless, these results are enough to pass to the limit equation (5). In a second step, we identify the limit $\nabla_y \mathcal{A}_0^M$ with $a(u_0^M) \nabla_y u_0^M$, see Section 5, where \mathcal{A}_0 denotes the two-scale limit of $\mathcal{A}(u_\epsilon^M)$. This implies the validity of the macroscopic problem (3), as well as the strong convergence of u_ϵ^M in the two-scale sense. The uniqueness of the weak solution follows by similar arguments as the uniqueness of the microscopic model, see Section 6. From the uniqueness, we obtain the convergence of the whole sequence.

4 Convergence results

In this section, we derive the convergence results for the sequence u_ϵ^\pm , u_ϵ^M , and $\mathcal{A}(u_\epsilon^M)$ based on the a priori estimates from Proposition 1 and two-scale compactness results. First of all, let us repeat the definition of the two-scale convergence for thin heterogeneous domains, see [2]:

Definition 1. We say a sequence $v_\epsilon^M \in L^2((0, T) \times \Omega_\epsilon^M)$ converges in the two-scale sense to a limit function $v_0^M \in L^2((0, T) \times \Sigma \times Z)$, if for all $\phi \in L^2((0, T) \times \Sigma, C_\#^\infty(Z))$ it holds that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} v_\epsilon^M(t, x) \phi \left(t, \bar{x}, \frac{x}{\epsilon} \right) dx dt = \int_0^T \int_{\Sigma} \int_Z v_0^M(t, \bar{x}, y) \phi(t, \bar{x}, y) dy d\bar{x} dt.$$

We say the sequence converges strongly in the two-scale sense, if additionally it holds that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \|v_\epsilon^M\|_{L^2((0,T) \times \Omega_\epsilon^M)} = \|v_0^M\|_{L^2((0,T) \times \Sigma \times Z)}.$$

Based on the a priori estimates in Proposition (1), we obtain the following convergence results:

Proposition 2. *Let u_ϵ be the solution of Problem (1).*

(i) *For the sequences in the bulk-domains u_ϵ^\pm , there exist $u_0^\pm \in L^2((0,T), H^1(\Omega^\pm))$ such that up to a subsequence it holds for $\frac{1}{2} < \beta < 1$*

$$\begin{aligned} u_\epsilon^\pm &\rightarrow u_0^\pm && \text{in } L^2((0,T), H^\beta(\Omega^\pm)), \\ \nabla u_\epsilon^\pm &\rightarrow \nabla u_0^\pm && \text{weakly in } L^2((0,T) \times \Omega^\pm). \end{aligned}$$

(ii) *For the sequence in the thin layer u_ϵ^M , there exists $u_0^M \in L^2((0,T) \times \Sigma, H_\#^1(Z))$ such that up to a subsequence it holds that*

$$\begin{aligned} u_\epsilon^M &\rightarrow u_0^M && \text{in the two-scale sense,} \\ \epsilon \nabla u_\epsilon^M &\rightarrow \nabla_y u_0^M && \text{in the two-scale sense.} \end{aligned}$$

Further, u_0^M is constant on S^\pm , and for almost every $(t, \bar{x}, \bar{y}) \in (0,T) \times \Sigma \times Y$ it holds that

$$u_0^\pm|_\Sigma(t, \bar{x}, 0) = u_0^M|_{S^\pm}(t, \bar{x}, \bar{y}, \pm 1).$$

(iii) *For the sequence of the Kirchhoff-transformation $\mathcal{A}(u_\epsilon^M)$ there exists $\mathcal{A}_0^M \in L^2((0,T) \times \Sigma, H_\#^1(Z))$, such that up to a subsequence it holds that*

$$\begin{aligned} \mathcal{A}(u_\epsilon^M) &\rightarrow \mathcal{A}_0^M && \text{in the two-scale sense,} \\ \epsilon \nabla \mathcal{A}(u_\epsilon^M) &\rightarrow \nabla_y \mathcal{A}_0^M && \text{in the two-scale sense.} \end{aligned}$$

Further, \mathcal{A}_0^M is constant on S^\pm , and for almost every $(t, \bar{x}, \bar{y}) \in (0,T) \times \Sigma \times Y$ it holds that

$$\mathcal{A}(u_0^\pm|_\Sigma)(t, \bar{x}, 0) = \mathcal{A}_0^M|_{S^\pm}(t, \bar{x}, \bar{y}, \pm 1).$$

Proof. Part (i) follows directly from the a priori estimates in Proposition 1 and the Aubin-Lions Lemma, see [14]. For the convergences in (ii) we use again Proposition 1 and [2, Proposition 4.4(ii)]. For the boundary condition $u_0^\pm|_\Sigma = u_0^M|_{S^\pm}$ see [2, Proposition 2.1]. By the same argument as above, we obtain the convergences in (iii) by using Corollary 1. For the boundary condition $\mathcal{A}(u_0^\pm|_\Sigma) = \mathcal{A}_0^M|_{S^\pm}$ we use $u_0^\pm|_\Sigma = u_0^M|_{S^\pm}$, the continuity of the operator \mathcal{A} , and the strong convergence of $u_\epsilon^\pm|_\Sigma$, which follows from the strong convergence of u_ϵ^\pm in $L^2((0,T), H^\beta(\Omega^\pm))$ and the trace theorem. \square

Now, we pass to the limit $\epsilon \rightarrow 0$ in the variational equation (2) for suitable test functions. For this, we define the following function spaces, see also [15]:

$$\begin{aligned} V_0 &:= \{ \phi = (\phi^+, \phi^M, \phi^-) \in H^1(\Omega^+) \times L^2(\Sigma, H_\#^1(Z)) \times H^1(\Omega^-) : \phi_0^\pm|_\Sigma = \phi_0^M|_{S^\pm} \}, \\ V_0^\infty &:= \left\{ \phi = (\phi^+, \phi^M, \phi^-) \in C^\infty(\overline{\Omega^+}) \times C_0^\infty(\Sigma, C_\#^\infty(\overline{Z})) \times C^\infty(\overline{\Omega^-}) : \phi_0^\pm|_\Sigma = \phi_0^M|_{S^\pm} \right\} \subset V_0. \end{aligned}$$

The space V_0^∞ is dense in V_0 . For $\phi \in C_0^\infty([0,T], V_0^\infty)$ we choose as a test function in (2)

$$\phi_\epsilon(t, x) := \begin{cases} \phi^+(t, x) & \text{for } (t, x) \in (0, T) \times \Omega^+, \\ \phi^M(t, \bar{x}, \frac{x}{\epsilon}) & \text{for } (t, x) \in (0, T) \times \Omega_\epsilon^M, \\ \phi^-(t, x) & \text{for } (t, x) \in (0, T) \times \Omega^-. \end{cases} \quad (4)$$

Then, Proposition 2 implies for $\epsilon \rightarrow 0$

$$\begin{aligned}
& \sum_{\pm} \left[\int_0^T \int_{\Omega^\pm} \nabla u_0^\pm \cdot \nabla \phi^\pm dx dt - \int_0^T \int_{\Omega^\pm} u_0^\pm \partial_t \phi^\pm dx dt \right] \\
& \quad - \int_0^T \int_{\Sigma} \int_Z u_0^M \partial_t \phi^M dy d\bar{x} dt + \int_0^T \int_{\Sigma} \int_Z D(y) \nabla_y \mathcal{A}_0^M \cdot \nabla_y \phi^M dy d\bar{x} dt \\
& = \sum_{\pm} \left[\int_0^T \int_{\Omega^\pm} f^\pm(u_0^\pm) \phi^\pm dx dt + \int_{\Omega^\pm} u_i^\pm \phi^\pm(0) dx \right] \\
& \quad + \int_0^T \int_{\Sigma} \int_Z g_0^M \phi^M dy d\bar{x} dt + \int_{\Sigma} \int_Z u_i^M \phi^M(0) dy d\bar{x}
\end{aligned} \tag{5}$$

for all $\phi \in C_0^\infty([0, T], V_0)$.

Remark 2. We have $\partial_t u_0^M \in L^2((0, T) \times \Sigma, \mathcal{H}_{S^\pm}^1(Z)')$ and a similar result holds for $\partial_t u_0^\pm$.

The crucial point is the identification of \mathcal{A}_0^M . We will show, that $\mathcal{A}_0^M = \mathcal{A}(u_0^M)$. This will be done in the next section.

5 Identification of the limit \mathcal{A}_0^M

The aim of this section is the identification of the limit \mathcal{A}_0^M with $\mathcal{A}(u_0^M)$. Therefore, we combine methods from [4] and [5]. In [4] a stationary monotone operator was considered. This problem was extended to a non-stationary single phase flow problem in [16]. In [5] a perforated domain was considered for a two-phase flow, and they use directly the macroscopic model which was derived by a formal asymptotic expansion. In [4] and [16] the macroscopic limit function is only depending on the macroscopic variable. Hence, we have to extend the methods to our case.

Let us choose $\eta_0 = (\eta_0^+, \eta_0^M, \eta_0^-) \in C^\infty([0, T], V_0^\infty)$ and η_ϵ in the same way as ϕ_ϵ in (4). Especially, we have $\eta_\epsilon^M(t, x) = \eta_0^M(t, \bar{x}, \frac{x}{\epsilon})$ and $\eta_\epsilon^M|_{S_\epsilon^\pm} = \eta_0^\pm|_\Sigma$. For fixed $t \in [0, T]$, we define $w_\epsilon^M \in \mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M)$ as the unique weak solution of the problem

$$\begin{aligned}
-\epsilon \nabla \cdot \left(D \left(\frac{x}{\epsilon} \right) \nabla w_\epsilon^M \right) &= \frac{1}{\epsilon} (u_\epsilon^M - \eta_\epsilon^M) && \text{in } \Omega_\epsilon^M, \\
w_\epsilon^M &= 0 && \text{on } S_\epsilon^\pm, \\
-\epsilon D \left(\frac{x}{\epsilon} \right) \nabla w_\epsilon^M \cdot \nu &= 0 && \text{on } \partial \Omega_\epsilon^M \setminus (S_\epsilon^+ \cup S_\epsilon^-).
\end{aligned} \tag{6}$$

For the space $\mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M)$ we have the following Poincaré-inequality:

$$\|\phi_\epsilon^M\|_{L^2(\Omega_\epsilon^M)} \leq C \epsilon \|\nabla \phi_\epsilon^M\|_{L^2(\Omega_\epsilon^M)} \quad \text{for all } \phi_\epsilon^M \in \mathcal{H}_{S_\epsilon^\pm}^1(\Omega_\epsilon^M). \tag{7}$$

Hence, the Lax-Milgram Lemma implies the existence of a unique weak solution w_ϵ^M , such that

$$\frac{1}{\sqrt{\epsilon}} \|w_\epsilon^M\|_{L^2(\Omega_\epsilon^M)} + \sqrt{\epsilon} \|\nabla w_\epsilon^M\|_{L^2(\Omega_\epsilon^M)} \leq \frac{C}{\sqrt{\epsilon}} \|u_\epsilon^M - \eta_\epsilon^M\|_{L^2(\Omega_\epsilon^M)} \leq C. \tag{8}$$

This implies the existence of $w_0^M \in L^2(\Sigma, H_{\#}^1(Z))$ such that up to a subsequence

$$\begin{aligned}
w_\epsilon^M &\rightarrow w_0^M && \text{in the two-scale sense,} \\
\epsilon \nabla w_\epsilon^M &\rightarrow \nabla_y w_0^M && \text{in the two-scale sense,}
\end{aligned}$$

and w_0^M is the unique weak solution of

$$\begin{aligned} -\nabla_y \cdot (D(y)\nabla_y w_0^M) &= u_0^M - \eta_0^M && \text{in } Z, \\ w_0^M &= 0 && \text{on } S^\pm, \\ w_0^M &\text{ is } Y\text{-periodic.} \end{aligned} \quad (9)$$

From (6), the definition of \mathcal{A} , (A4), and the continuity of u_ϵ across S_ϵ^\pm we obtain

$$\begin{aligned} &\epsilon \int_0^T \int_{\Omega_\epsilon^M} D\left(\frac{x}{\epsilon}\right) \nabla [\mathcal{A}(u_\epsilon^M) - \mathcal{A}(\eta_\epsilon^M)] \cdot \nabla w_\epsilon^M dx dt \\ &= \epsilon \int_0^T \int_{\partial\Omega_\epsilon^M} [\mathcal{A}(u_\epsilon^M) - \mathcal{A}(\eta_\epsilon^M)] D\left(\frac{x}{\epsilon}\right) \nabla w_\epsilon^M \cdot \nu d\sigma dt \\ &\quad + \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} [\mathcal{A}(u_\epsilon^M) - \mathcal{A}(\eta_\epsilon^M)] [u_\epsilon^M - \eta_\epsilon^M] dx dt \\ &\geq \epsilon \sum_{\pm} \int_0^T \int_{S_\epsilon^\pm} [\mathcal{A}(u_\epsilon^\pm) - \mathcal{A}(\eta_0^\pm)] D\left(\frac{x}{\epsilon}\right) \nabla w_\epsilon^M \cdot \nu d\sigma dt \\ &\quad + \frac{a_0}{\epsilon} \|u_\epsilon^M - \eta_\epsilon^M\|_{L^2((0,T)\times\Omega_\epsilon^M)}^2. \end{aligned} \quad (10)$$

Now, we extend the function u_ϵ^\pm to a function \tilde{u}_ϵ^\pm defined on the whole domain $\Sigma \times \mathbb{R}$, such that

$$\|\tilde{u}_\epsilon^\pm\|_{H^1(\Sigma \times \mathbb{R})} \leq C \|u_\epsilon^\pm\|_{H^1(\Omega^\pm)}. \quad (11)$$

We define for $(t, x) \in (0, T) \times \Omega_\epsilon^M$

$$\begin{aligned} \tilde{v}_\epsilon^M(t, x) &:= \frac{\epsilon + x_n}{2\epsilon} \tilde{u}_\epsilon^+(t, x) + \frac{\epsilon - x_n}{2\epsilon} \tilde{u}_\epsilon^-(t, x), \\ \tilde{\eta}_\epsilon^M(t, x) &:= \frac{\epsilon + x_n}{2\epsilon} \eta_0^+(t, \bar{x}) + \frac{\epsilon - x_n}{2\epsilon} \eta_0^-(t, \bar{x}), \end{aligned}$$

and for $(t, \bar{x}, y) \in (0, T) \times \Sigma \times Z$

$$\begin{aligned} \tilde{v}_0^M(t, \bar{x}, y) &:= \frac{1 + y_n}{2} u_0^+(t, \bar{x}, 0) + \frac{1 - y_n}{2} u_0^-(t, \bar{x}, 0), \\ \tilde{\eta}_0^M(t, \bar{x}, y) &:= \frac{1 + y_n}{2} \eta_0^+(t, \bar{x}) + \frac{1 - y_n}{2} \eta_0^-(t, \bar{x}). \end{aligned}$$

Lemma 1. *It holds that*

$$\begin{aligned} \tilde{v}_\epsilon^M &\rightarrow v_0^M && \text{strongly in the two-scale sense,} \\ \tilde{\eta}_\epsilon^M &\rightarrow \tilde{\eta}_0^M && \text{strongly in the two-scale sense.} \end{aligned}$$

Additionally, we have

$$\tilde{u}_\epsilon^\pm|_{\Omega_\epsilon^M} \rightarrow u_0^\pm|_\Sigma \quad \text{strongly in the two-scale sense in } L^2((0, T) \times \Omega_\epsilon^M).$$

Proof. From the mean value theorem and the a priori estimates for u_ϵ^\pm from Proposition 1 combined with (11) we get

$$\frac{1}{\sqrt{\epsilon}} \|\tilde{u}_\epsilon^\pm - u_\epsilon^\pm(t, \bar{x}, 0)\|_{L^2((0,T)\times\Omega_\epsilon^M)} \leq C\sqrt{\epsilon}.$$

Together with strong convergence of $u_\epsilon^\pm|_\Sigma$ from Proposition 2(i), we immediately obtain the strong two-scale convergence of $\tilde{u}_\epsilon^\pm|_{\Omega_\epsilon^M}$. Further, it holds that

$$\begin{aligned} \frac{1}{\sqrt{\epsilon}} \left\| \tilde{v}_\epsilon^M - \tilde{v}_0^M\left(t, \bar{x}, \frac{x}{\epsilon}\right) \right\|_{L^2((0,T)\times\Omega_\epsilon^M)} &\leq \frac{1}{\sqrt{\epsilon}} \sum_{\pm} \left\| \frac{\epsilon \pm x_n}{\epsilon} [\tilde{u}_\epsilon^\pm - u_0^\pm|_\Sigma] \right\|_{L^2((0,T)\times\Omega_\epsilon^M)} \\ &\leq C\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} \|u_\epsilon^\pm(t, \bar{x}, 0) - u_0^\pm|_\Sigma\|_{L^2((0,T)\times\Omega_\epsilon^M)}. \end{aligned}$$

Using again the strong convergence of $u_\epsilon^\pm|_\Sigma$, this implies the strong two-scale convergence of \tilde{v}_ϵ^M . The strong convergence of $\tilde{\eta}_\epsilon^M$ is obvious, due to its regularity properties. \square

By integration by parts, we obtain

$$\begin{aligned} & \epsilon \sum_{\pm} \int_0^T \int_{S_\epsilon^\pm} [\mathcal{A}(u_\epsilon^\pm) - \mathcal{A}(\eta_0^\pm)] D\left(\frac{x}{\epsilon}\right) \nabla w_\epsilon^M \cdot \nu d\sigma dt \\ &= \epsilon \sum_{\pm} \int_0^T \int_{S_\epsilon^\pm} [\mathcal{A}(\tilde{v}_\epsilon^M) - \mathcal{A}(\tilde{\eta}_\epsilon^M)] D\left(\frac{x}{\epsilon}\right) \nabla w_\epsilon^M \cdot \nu d\sigma dt \\ &= \epsilon \int_0^T \int_{\Omega_\epsilon^M} \nabla [\mathcal{A}(\tilde{v}_\epsilon^M) - \mathcal{A}(\tilde{\eta}_\epsilon^M)] \cdot D\left(\frac{x}{\epsilon}\right) \nabla w_\epsilon^M dx dt \\ &\quad - \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} [\mathcal{A}(\tilde{v}_\epsilon^M) - \mathcal{A}(\tilde{\eta}_\epsilon^M)] (u_\epsilon^M - \eta_\epsilon^M) dx dt =: A_\epsilon^1 + A_\epsilon^2. \end{aligned}$$

For the first term on the right-hand side we use

$$\begin{aligned} \nabla \mathcal{A}(\tilde{v}_\epsilon^M) &= a(\tilde{v}_\epsilon^M) \sum_{\pm} \left[\frac{\epsilon \pm x_n}{2\epsilon} \nabla \tilde{u}_\epsilon^\pm \pm \frac{1}{2\epsilon} \tilde{u}_\epsilon^\pm e_n \right], \\ \nabla \mathcal{A}(\tilde{\eta}_\epsilon^M) &= a(\tilde{\eta}_\epsilon^M) \sum_{\pm} \left[\frac{\epsilon \pm x_n}{2\epsilon} \nabla_{\tilde{x}} \eta_0^\pm \pm \frac{1}{2\epsilon} \eta_0^\pm e_n \right], \end{aligned}$$

to obtain

$$\begin{aligned} A_\epsilon^1 &= \sum_{\pm} \left[\frac{1}{2\epsilon} \int_0^T \int_{\Omega_\epsilon^M} \pm [a(\tilde{v}_\epsilon^M) \tilde{u}_\epsilon^\pm e_n - a(\tilde{\eta}_\epsilon^M) \eta_0^\pm e_n] \cdot D\left(\frac{x}{\epsilon}\right) \epsilon \nabla w_\epsilon^M dx dt \right. \\ &\quad \left. + \int_0^T \int_{\Omega_\epsilon^M} \left[a(\tilde{v}_\epsilon^M) \frac{\epsilon \pm x_n}{2\epsilon} \nabla \tilde{u}_\epsilon^\pm - a(\tilde{\eta}_\epsilon^M) \frac{\epsilon \pm x_n}{2\epsilon} \nabla_{\tilde{x}} \eta_0^\pm \right] \cdot D\left(\frac{x}{\epsilon}\right) \epsilon \nabla w_\epsilon^M dx dt. \right] \end{aligned}$$

The second term is of order $\sqrt{\epsilon}$ and vanishes for $\epsilon \rightarrow 0$. Together with the strong convergences from Lemma 1 and the weak convergence of $\epsilon \nabla w_\epsilon^M$ we obtain

$$\begin{aligned} A_\epsilon^1 &\xrightarrow{\epsilon \rightarrow 0} \pm \frac{1}{2} \sum_{\pm} \int_0^T \int_{\Sigma} \int_Z [a(\tilde{v}_0^M) u_0^\pm|_\Sigma e_n - a(\tilde{\eta}_0^M) \eta_0^\pm e_n] \cdot D(y) \nabla_y w_0^M dy d\bar{x} dt \\ &= \int_0^T \int_{\Sigma} \int_Z \nabla_y [\mathcal{A}(\tilde{v}_0^M) - \mathcal{A}(\tilde{\eta}_0^M)] \cdot D(y) \nabla_y w_0^M dy d\bar{x} dt. \end{aligned}$$

Further, we have, due to the strong convergence of \tilde{v}_ϵ^M and $\tilde{\eta}_\epsilon^M$,

$$A_\epsilon^2 \xrightarrow{\epsilon \rightarrow 0} - \int_0^T \int_{\Sigma} \int_Z [\mathcal{A}(\tilde{v}_0^M) - \mathcal{A}(\tilde{\eta}_0^M)] (u_0^M - \eta_0^M) dy d\bar{x} dt.$$

Altogether, we obtain with (9)

$$\begin{aligned} & \epsilon \sum_{\pm} \int_0^T \int_{S_\epsilon^\pm} [\mathcal{A}(u_\epsilon^\pm) - \mathcal{A}(\eta_0^\pm)] D\left(\frac{x}{\epsilon}\right) \nabla w_\epsilon^M \cdot \nu d\sigma dt \\ &\xrightarrow{\epsilon \rightarrow 0} \int_0^T \int_{\Sigma} \int_Z \nabla_y [\mathcal{A}(\tilde{v}_0^M) - \mathcal{A}(\tilde{\eta}_0^M)] \cdot D(y) \nabla_y w_0^M dy d\bar{x} dt \\ &\quad - \int_0^T \int_{\Sigma} \int_Z [\mathcal{A}(\tilde{v}_0^M) - \mathcal{A}(\tilde{\eta}_0^M)] (u_0^M - \eta_0^M) dy d\bar{x} dt \\ &= \int_0^T \int_{\Sigma} \int_{\partial Z} [\mathcal{A}(\tilde{v}_0^M) - \mathcal{A}(\tilde{\eta}_0^M)] D(y) \nabla_y w_0^M \cdot \nu d\sigma_y d\bar{x} dt \\ &= \sum_{\pm} \int_0^T \int_{\Sigma} \int_{S_\pm} [\mathcal{A}(u_0^\pm|_\Sigma) - \mathcal{A}(\eta_0^\pm)] D(y) \nabla_y w_0^M \cdot \nu d\sigma_y d\bar{x} dt. \end{aligned}$$

Altogether, we obtain for (10) for $\epsilon \rightarrow 0$ by using the boundary condition $u_0^\pm|_\Sigma = u_0^M|_{S^\pm}$

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0} \frac{a_0}{\epsilon} \|u_\epsilon^M - \eta_\epsilon^M\|_{L^2((0,T) \times \Omega_\epsilon^M)} \\
&\leq - \int_0^T \int_\Sigma \int_{\partial Z} [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] D(y) \nabla_y w_0^M \cdot \nu d\sigma_y d\bar{x} dt \\
&\quad + \lim_{\epsilon \rightarrow 0} \epsilon \int_0^T \int_{\Omega_\epsilon^M} D\left(\frac{x}{\epsilon}\right) \nabla [\mathcal{A}(u_\epsilon^M) - \mathcal{A}(\eta_\epsilon^M)] \cdot \nabla w_\epsilon^M dx dt.
\end{aligned} \tag{12}$$

Let us pass to the limit $\epsilon \rightarrow 0$ in the second term on the right-hand side. The problem is that we only have weak convergences for the sequences $\nabla \mathcal{A}(u_\epsilon^M)$ and ∇w_ϵ^M . Therefore, we use the that u_ϵ^M is a weak solution of Problem (1). First of all, the strong convergence of η_ϵ^M and its gradient implies

$$\epsilon \int_0^T \int_{\Omega_\epsilon^M} D\left(\frac{x}{\epsilon}\right) \nabla \mathcal{A}(\eta_\epsilon^M) \cdot \nabla w_\epsilon^M dx dt \xrightarrow{\epsilon \rightarrow 0} \int_0^T \int_\Sigma \int_Z D(y) \nabla_y \mathcal{A}(\eta_0^M) \cdot \nabla_y w_0^M dy d\bar{x} dt.$$

Further, (2) implies (see also Remark 1)

$$\begin{aligned}
&\epsilon \int_0^T \int_{\Omega_\epsilon^M} D\left(\frac{x}{\epsilon}\right) \nabla \mathcal{A}(u_\epsilon^M) \cdot \nabla w_\epsilon^M dx dt \\
&= -\frac{1}{\epsilon} \int_0^T \langle \partial_t u_\epsilon^M, w_\epsilon^M \rangle_{\mathcal{H}^1_{S^\pm}(\Omega_\epsilon^M)', \mathcal{H}^1_{S^\pm}(\Omega_\epsilon^M)} dt + \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} g_\epsilon^M w_\epsilon^M dx dt.
\end{aligned}$$

For the first term on the right-hand side we can write with (6)

$$\begin{aligned}
&-\frac{1}{\epsilon} \int_0^T \langle \partial_t u_\epsilon^M, w_\epsilon^M \rangle_{\mathcal{H}^1_{S^\pm}(\Omega_\epsilon^M)', \mathcal{H}^1_{S^\pm}(\Omega_\epsilon^M)} dt \\
&= \frac{\epsilon}{2} \left\| \sqrt{D\left(\frac{x}{\epsilon}\right)} \nabla w_\epsilon^M(0) \right\|_{L^2(\Omega_\epsilon^M)}^2 - \frac{\epsilon}{2} \left\| \sqrt{D\left(\frac{x}{\epsilon}\right)} \nabla w_\epsilon^M(T) \right\|_{L^2(\Omega_\epsilon^M)}^2 - \frac{1}{\epsilon} \int_0^T \int_{\Omega_\epsilon^M} \partial_t \eta_\epsilon^M w_\epsilon^M dx dt,
\end{aligned}$$

and using again (6) for $t = 0$, we get

$$\begin{aligned}
\frac{\epsilon}{2} \left\| \sqrt{D\left(\frac{x}{\epsilon}\right)} \nabla w_\epsilon^M(0) \right\|_{L^2(\Omega_\epsilon^M)}^2 &= \frac{1}{2\epsilon} \int_{\Omega_\epsilon^M} (u_{\epsilon,i}^M - \eta_\epsilon^M(0)) w_\epsilon^M(0) dx dt \\
&\xrightarrow{\epsilon \rightarrow 0} \frac{1}{2} \int_\Sigma \int_Z (u_i^M - \eta_0^M(0)) w_0^M(0) dy d\bar{x},
\end{aligned}$$

where we used the strong convergence of $u_{\epsilon,i}^M$ from Assumption (A5). Together with the strong two-scale convergence of g_ϵ^M , see Assumption (A2), we obtain for $\epsilon \rightarrow 0$

$$\begin{aligned}
&\lim_{\epsilon \rightarrow 0} \epsilon \int_0^T \int_{\Omega_\epsilon^M} D\left(\frac{x}{\epsilon}\right) \nabla \mathcal{A}(u_\epsilon^M) \cdot \nabla w_\epsilon^M dx dt \\
&\leq \frac{1}{2} \int_\Sigma \int_Z (u_i^M - \eta_0^M(0)) w_0^M(0) dy d\bar{x} \\
&\quad + \int_0^T \int_\Sigma \int_Z g_0 w_0^M dy d\bar{x} dt - \int_0^T \int_\Sigma \int_Z \partial_t \eta_0^M w_0^M dy d\bar{x} dt.
\end{aligned}$$

Hence, we obtain from (12)

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0} \frac{a_0}{\epsilon} \|u_\epsilon^M - \eta_\epsilon^M\|_{L^2((0,T) \times \Omega_\epsilon^M)} \\
&\leq - \int_0^T \int_\Sigma \int_{\partial Z} [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] D(y) \nabla_y w_0^M \cdot \nu d\sigma_y d\bar{x} dt \\
&\quad - \int_0^T \int_\Sigma \int_Z D(y) \nabla_y \mathcal{A}(\eta_0^M) \cdot \nabla_y w_0^M dy d\bar{x} dt + \frac{1}{2} \int_\Sigma \int_Z (u_i^M - \eta_0^M(0)) w_0^M(0) dy d\bar{x} \\
&\quad + \int_0^T \int_\Sigma \int_Z g_0 w_0^M dy d\bar{x} dt - \int_0^T \int_\Sigma \int_Z \partial_t \eta_0^M w_0^M dy d\bar{x} dt.
\end{aligned}$$

Choosing in (5) the test function $\phi_0^M = w_0^M$ and $\phi_0^\pm = 0$, we get

$$\begin{aligned}
\int_0^T \int_\Sigma \int_Z g_0^M w_0^M dy d\bar{x} dt &= \int_0^T \langle \partial_t u_0^M, w_0^M \rangle_{L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z)'), L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z))} dt \\
&\quad + \int_0^T \int_\Sigma \int_Z D(y) \nabla_y \mathcal{A}_0^M \cdot \nabla_y w_0^M dy d\bar{x} dt.
\end{aligned}$$

Under the use of (here we use (9))

$$\begin{aligned}
&- \int_0^T \int_\Sigma \int_{\partial Z} [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] D(y) \nabla_y w_0^M \cdot \nu d\sigma_y d\bar{x} dt \\
&= \int_0^T \int_\Sigma \int_Z D(y) \nabla_y [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] \cdot \nabla_y w_0^M dy d\bar{x} dt \\
&\quad - \int_0^T \int_\Sigma \int_Z [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] (u_0^M - \eta_0^M) dy d\bar{x} dt.
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \langle \partial_t u_0^M, w_0^M \rangle_{L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z)'), L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z))} dt \\
&= \frac{1}{2} \int_\Sigma \int_Z (u_0^M(T) - \eta_0^M(T)) w_0^M(T) dy d\bar{x} - \frac{1}{2} \int_\Sigma \int_Z (u_i^M - \eta_0^M(0)) w_0^M(0) dy d\bar{x} \\
&\quad + \int_0^T \int_\Sigma \int_Z \partial_t \eta_0^M w_0^M dy d\bar{x} dt,
\end{aligned}$$

we obtain

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0} \frac{a_0}{2} \|u_\epsilon^M - \eta_\epsilon^M\|_{L^2((0,T) \times \Omega_\epsilon^M)}^2 \leq \frac{1}{2} \int_\Sigma \int_Z (u_0^M(T) - \eta_0^M(T)) w_0^M(T) dy d\bar{x} \\
&\quad - \int_0^T \int_\Sigma \int_Z [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] (u_0^M - \eta_0^M) dy d\bar{x} dt \\
&\quad + \int_0^T \int_\Sigma \int_Z D(y) \nabla_y [\mathcal{A}_0^M - \mathcal{A}(u_0^M)] \cdot \nabla_y w_0^M dy d\bar{x} dt.
\end{aligned}$$

Since $\mathcal{A}_0^M|_{S^\pm} = \mathcal{A}(u_0^M)|_{S^\pm}$, we have that $\mathcal{A}_0^M - \mathcal{A}(u_0^M)$ is an admissible test function for the weak formulation of (9). Hence, we get

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0} \frac{a_0}{2} \|u_\epsilon^M - \eta_\epsilon^M\|_{L^2((0,T) \times \Omega_\epsilon^M)}^2 \leq \frac{1}{2} \int_\Sigma \int_Z (u_0^M(T) - \eta_0^M(T)) w_0^M(T) dy d\bar{x} \\
&\quad - \int_0^T \int_\Sigma \int_Z [\mathcal{A}(u_0^M) - \mathcal{A}(\eta_0^M)] (u_0^M - \eta_0^M) dy d\bar{x} dt \quad (13) \\
&\quad + \int_0^T \int_\Sigma \int_Z [\mathcal{A}_0^M - \mathcal{A}(u_0^M)] (u_0^M - \eta_0^M) dy d\bar{x} dt.
\end{aligned}$$

By density, we can choose $\eta_0^M(t, \bar{x}, y) = u_0^M(t, \bar{x}, y) + \lambda\phi(t, \bar{x}, y)$ for $\lambda \in \mathbb{R}$ and $\phi \in C_0^\infty((0, T) \times \Sigma \times Z)$ and obtain

$$0 \leq \int_0^T \int_\Sigma \int_Z [\mathcal{A}(u_0^M) - \mathcal{A}(u_0^M + \lambda\phi)] \lambda\phi dy d\bar{x} dt + \int_0^T \int_\Sigma \int_Z [\mathcal{A}_0^M - \mathcal{A}(u_0^M)] \lambda\phi dy d\bar{x} dt.$$

Hence, dividing by $\lambda \neq 0$, we obtain for $\lambda \rightarrow 0^+$ and $\lambda \rightarrow 0^-$

$$0 = \int_0^T \int_\Sigma \int_Z [\mathcal{A}_0^M - \mathcal{A}(u_0^M)] \phi dy d\bar{x} dt,$$

and this implies :

Proposition 3.

$$\mathcal{A}_0^M = \mathcal{A}(u_0^M).$$

Corollary 2. *We have*

$$u_\epsilon^M \rightarrow u_0^M \quad \text{strongly in the two-scale sense.}$$

Proof. In (13), we choose $\eta_0^M = u_k^M \in C^\infty([0, T], V_0^\infty)$, such that

$$u_k^M \rightarrow u_0^M \quad \text{in } L^2((0, T), L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z))) \cap H^1((0, T), L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z)')).$$

For $\epsilon \rightarrow 0$ we get

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{a_0}{\epsilon} \left\| u_\epsilon^M - u_k^M \left(t, \bar{x}, \frac{x}{\epsilon} \right) \right\|_{L^2((0, T) \times \Omega_\epsilon^M)}^2 \\ & \leq - \int_0^T \int_\Sigma \int_Z [\mathcal{A}(u_0^M) - \mathcal{A}(u_k^M)] (u_0^M - u_k^M) dy d\bar{x} dt \\ & \quad + \frac{1}{2} \int_\Sigma \int_Z (u_0^M(T) - u_k^M(T)) w_0^M(T) dy d\bar{x} =: \Delta_k. \end{aligned}$$

The first term in Δ_k vanishes for $k \rightarrow \infty$, since $u_k^M \rightarrow u_0^M$ in $L^2((0, T) \times \Sigma \times Z)$. For the second term we use the continuity of the embedding $L^2((0, T), L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z))) \cap H^1((0, T), L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z)'))$ into $C^0([0, T], L^2(\Sigma, \mathcal{H}_{S^\pm}^1(Z)'))$, see [17, Lemma 7.1], to obtain $\Delta_k \rightarrow 0$ for $k \rightarrow \infty$. By the lower semicontinuity of the two-scale convergence and the triangle inequality, we obtain

$$\begin{aligned} \|u_0^M\|_{L^2((0, T) \times \Sigma \times Z)} & \leq \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \|u_\epsilon^M\|_{L^2((0, T) \times \Omega_\epsilon^M)} \\ & \leq \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\sqrt{\epsilon}} \left\| u_\epsilon^M - u_k^M \left(t, \bar{x}, \frac{x}{\epsilon} \right) \right\|_{L^2((0, T) \times \Omega_\epsilon^M)} + \frac{1}{\sqrt{\epsilon}} \left\| u_k^M \left(t, \bar{x}, \frac{x}{\epsilon} \right) \right\|_{L^2((0, T) \times \Omega_\epsilon^M)} \right) \\ & \leq \sqrt{\Delta_k} + \|u_k^M\|_{L^2((0, T) \times \Sigma \times Z)} \xrightarrow{k \rightarrow \infty} \|u_0^M\|_{L^2((0, T) \times \Sigma \times Z)}. \end{aligned}$$

□

Proof of Theorem 1. Altogether, we obtain the claim of Theorem 1, except the uniqueness. This can be established in a similar way as the uniqueness of the microscopic model, see Section 6. □

6 Uniqueness of the microscopic model

In this section, we prove uniqueness for the microscopic problem (1). However, to simplify the notation we omit the ϵ -dependence of the problem and restrict ourselves to the case when the domain is separated into two subdomains. Further, we consider a more general

nonlinear parabolic problem which includes our case. The method is based on [18], where methods from [19] and [20] are used. An overview over this topic can be found in [21]. While in [18] an equation of the form

$$\partial_t[b(u)] - \nabla \cdot (a(\nabla u, b(u)) + f(b(u)) = 0,$$

was considered, we only treat the case when b is the identity and a is additionally an elliptic operator (see Assumption (H2)). Additionally, we only consider the Hilbert-case. However, these assumptions are enough to apply the following results to our microscopic model, and make the proof much less technical. We cannot apply directly the results from [18] to our problem, since in our case the function a is not necessarily continuous across the interface between the two subdomains.

We consider a connected domain $\Omega \subset \mathbb{R}^n$ with Lipschitz-boundary, which is separated into two disjoint connected subdomains Ω_1 and Ω_2 , with $\Sigma := \text{int}(\overline{\Omega_1} \cap \overline{\Omega_2})$. Hence, we have

$$\Omega = \Omega_1 \cup \Sigma \cup \Omega_2.$$

Further, we assume that Ω_i for $i = 1, 2$ has a Lipschitz-boundary, especially, Σ is a Lipschitz-surface. Additionally, we make the assumption that Ω_i has positive measure, as well as Σ . Let us consider the following problem for $i = 1, 2$:

$$\begin{aligned} \partial_t u_i - \nabla \cdot (a_i(\nabla u_i, u_i)) &= f_i(u_i) && \text{in } (0, T) \times \Omega_i, \\ u_1 &= u_2 && \text{on } (0, T) \times \Sigma, \\ a_1(\nabla u_1, u_1) \cdot \nu &= a_2(\nabla u_2, u_2) \cdot \nu && \text{on } (0, T) \times \Sigma, \\ u_i(0) &= u_i^0 && \text{in } \Omega_i, \\ a_i(\nabla u_i, u_i) \cdot \nu &= 0 && \text{on } (0, T) \times \partial\Omega_i \setminus \Sigma. \end{aligned} \quad (14)$$

We denote a solution of Problem (14) by u , where $u = u_i$ on Ω_i . Let us denote the function space of Sobolev functions with zero traces on Σ by

$$\mathcal{H}_0^1(\Omega_i) := \{v_i \in H^1(\Omega_i) : v_i = 0 \text{ on } \Sigma\}.$$

We say that u is a weak solution of Problem (14), if $u \in L^2((0, T), H^1(\Omega))$ with $\partial_t u_i \in L^2((0, T), \mathcal{H}_0^1(\Omega_i)')$, and for all $\phi \in L^2((0, T), H^1(\Omega))$ with $\partial_t \phi \in L^2((0, T) \times \Omega)$ and $\phi(T) = 0$ it holds the following variational equation

$$\begin{aligned} \sum_{i=1}^2 - \int_0^T \int_{\Omega_i} u_i \partial_t \phi dx dt + \int_0^T \int_{\Omega_i} a_i(\nabla u_i, u_i) \cdot \nabla \phi dx dt \\ = \sum_{i=1}^2 \int_0^T \int_{\Omega_i} f_i(u_i) \phi dx dt + \int_{\Omega_i} u_i^0 \phi(0) dx. \end{aligned}$$

With $u^0 := u_i^0$ in Ω_i , we can write the above equation in the form

$$\int_0^T \int_{\Omega} [u^0 - u] \partial_t \phi dx dt + \sum_{i=1}^2 \int_0^T \int_{\Omega_i} a_i(\nabla u_i, u_i) \cdot \nabla \phi dx dt = \sum_{i=1}^2 \int_0^T \int_{\Omega_i} f_i(u_i) \phi dx dt. \quad (15)$$

Remark 3. We emphasize that $u_i \in C^0([0, T], \mathcal{H}_0^1(\Omega_i)')$, see e. g., [17, Lemma 7.1], and therefore we have $u_i(0) = u_i^0$ in $\mathcal{H}_0^1(\Omega_i)'$.

Assumptions on the data:

(H1) The function $f_i : [0, T] \times \overline{\Omega_i} \times \mathbb{R} \rightarrow \mathbb{R}$ for $i = 1, 2$ is continuous and uniformly Lipschitz-continuous with respect to the third variable.

(H2) For $i = 1, 2$ we have that $a_i : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is continuous and fulfills the following growth condition

$$\|a_i(p, w)\|^2 \leq C(1 + \|p\|^2 + |w|^2) \quad \text{for all } p \in \mathbb{R}^n, w \in \mathbb{R}.$$

There exists $a_0 > 0$, such that

$$a_i(p, w) \cdot p \geq a_0 \|p\|^2 \quad \text{for all } p \in \mathbb{R}^n, w \in \mathbb{R}.$$

Further, a_i is strictly monotone with respect to p , i. e., there exists $c_0 > 0$ such that for all $p_1, p_2 \in \mathbb{R}^n, w \in \mathbb{R}$ it holds that

$$[a_i(p_1, w) - a_i(p_2, w)] \cdot [p_1 - p_2] \geq c_0 \|p_1 - p_2\|^2.$$

a_i is Hölder-continuous with respect to w , i. e., there exists L_0 , such that for all $p \in \mathbb{R}^n, w_1, w_2 \in \mathbb{R}$ it holds that

$$\|a_i(p, w_1) - a_i(p, w_2)\|^2 \leq L_0 |w_1 - w_2| [1 + \|p\|^2 + |w_1|^2 + |w_2|^2].$$

(H3) It holds that $u^0 \in L^2(\Omega)$.

Theorem 2. *There exists at most one weak solution of Problem (14).*

The proof follows similar lines as in [18]. However, for the sake of completeness, we give its main ingredients. As mentioned above, we can avoid some technical problems, due to our additional assumption on a_i (and b). Let us start with the following lemma.

Lemma 2. *Let $\eta \in C^2(\mathbb{R})$ be a convex function with bounded first and second derivatives, and define*

$$q(z, z^0) := \int_{z^0}^z \eta'(\zeta - z^0) d\zeta \quad \text{for } z, z^0 \in \mathbb{R}. \quad (16)$$

If u is a weak solution of Problem (14) and $w \in H^1(\Omega)$, then for all nonnegative $\gamma \in C_0^\infty([0, T])$ it holds that

$$\begin{aligned} & \int_0^T \int_\Omega [q(u^0, w) - q(u, w)] \partial_t \gamma dx dt + \sum_{i=1}^2 \int_0^T \int_{\Omega_i} a_i(\nabla u_i, u_i) \cdot \nabla [\eta'(u - w)] \gamma dx dt \\ & \leq \sum_{i=1}^2 \int_0^T \int_{\Omega_i} f_i(u_i) \eta'(u_i - w) \gamma dx dt. \end{aligned} \quad (17)$$

Proof. We only give the main ideas of the proof. For more details see [18, Lemma 1(a)]. First of all, we define

$$\phi := \eta'(u - w) \gamma \in L^2((0, T), H^1(\Omega)).$$

However, this is not an admissible test function for (15). Therefore, we regularize with respect to time by defining for $t \in (0, T)$ and $h > 0$

$$\phi_h := \frac{1}{h} \int_t^{t+h} \phi(s) ds.$$

Here, we can extend the function u for $t < 0$ by the constant value u^0 . For $T < t$ we extend u by zero, what has no influence, due to the compact support of γ in $[0, T]$. Since η is convex, it holds for all $z_1, z_2, z^0 \in \mathbb{R}$ that

$$q(z_2, z^0) - q(z_1, z^0) = \int_{z_1}^{z_2} \eta'(\zeta - z^0) d\zeta \geq \eta'(z_1 - z^0) \cdot (z_2 - z_1).$$

This implies, together with the integration by parts formula for difference quotients,

$$\begin{aligned} \int_0^T \int_{\Omega} [u^0 - u(t)] \partial_t \phi_h dx dt &= \frac{1}{h} \int_0^T \int_{\Omega} \eta'(u(t) - w) \gamma(t) [u(t) - u(t-h)] dx dt \\ &\geq \frac{1}{h} \int_0^T \int_{\Omega} [q(u(t), w) - q(u(t-h), w)] \gamma(t) dx dt \\ &= - \int_0^T \int_{\Omega} q(u(t), w) \frac{1}{h} [\gamma(t+h) - \gamma(t)] dx dt. \end{aligned}$$

Hence,

$$\liminf_{h \rightarrow 0} - \int_0^T \int_{\Omega} u(t) \partial_t \phi_h dx dt \geq - \int_0^T \int_{\Omega} -q(u(t), w) \partial_t \gamma dx dt.$$

This gives the desired result. \square

We already know that $u_i \in C^0([0, T], \mathcal{H}_0^1(\Omega_i)')$, but this result is not enough and we need some kind of L^1 -convergence of $u(t)$ to u^0 for $t \rightarrow 0$.

Lemma 3. *Let u be a weak solution of the Problem (14). Then, there exists a subset $\mathcal{N} \subset (0, T)$ of measure zero, such that*

$$\lim_{t \rightarrow 0, t \notin \mathcal{N}} \int_{\Omega} (u(t) - u^0)^+ dx = 0,$$

where we define $(\cdot)^+ := \max\{0, \cdot\}$.

Proof. The proof follows similar lines as [18, Lemma 2]. Let η be a smooth and convex function with

$$\eta(z) = \begin{cases} 0 & \text{for } z \leq 0, \\ z - \frac{1}{2} & \text{for } z \geq 1. \end{cases}$$

and define for $\delta > 0$

$$\eta_{\delta}(z) := \delta \eta\left(\frac{z}{\delta}\right),$$

and denote by q_{δ} the function defined by (16) with respect to η_{δ} . Further, we approximate $u_0 \in L^2(\Omega)$ by smooth functions u_k^0 .

Now, we use η_{δ} , u_k^0 , and $\gamma = \alpha \in C_0^{\infty}([0, T])$ with $\alpha \geq 0$ in Lemma 2 to obtain (with the boundedness of η'_{δ})

$$\begin{aligned} - \int_0^T \alpha'(t) \int_{\Omega} [q_{\delta}(u, u_k^0) - q_{\delta}(u^0, u_k^0)] dx dt &\leq \sum_{i=1}^2 \int_0^T \alpha'(t) \int_{\Omega_i} f_i(u_i) \eta'_{\delta}(u_i - u^0) dx dt \\ &\leq \int_0^T \alpha(t) C(1 + \|u\|_{L^2(\Omega)}) dt =: \int_0^T \alpha(t) \theta(t) dt, \end{aligned} \tag{18}$$

with $\theta \in L^1((0, T))$ independent of k and δ . An elemental calculation shows for all $z, z^0 \in \mathbb{R}$

$$0 \leq q_{\delta}(z, z^0) \leq (z - z^0)^+ \leq \frac{\delta}{2} + q_{\delta}(z, z^0).$$

This implies together with the dominated convergence theorem of Lebesgue

$$\begin{aligned} \int_{\Omega} q_{\delta}(u^0, u_k^0) dx &\leq \int_{\Omega} (u^0 - u_k^0)^+ dx, \\ q_{\delta}(u, u_k^0) &\xrightarrow{\delta \rightarrow 0} (u - u_k^0)^+ \quad \text{in } L^1((0, T) \times \Omega). \end{aligned}$$

Hence, we obtain from (18)

$$-\int_0^T \alpha'(t) \int_{\Omega} [q_{\delta}(u, u_k^0) - (u^0 - u_k^0)^+] dx dt \leq \int_0^T \alpha(t) \theta(t) dt,$$

and for $k \rightarrow \infty$

$$-\int_0^T \alpha'(t) \int_{\Omega} (u - u^0)^+ dx dt \leq \int_0^T \alpha(t) \theta(t) dt.$$

Hence, there exists a set of measure zero $\mathcal{N} \subset (0, T)$, such that

$$\lim_{t \rightarrow 0, t \notin \mathcal{N}} \int_{\Omega} (u(t) - u^0)^+ dx = 0.$$

□

Proof of Theorem 2. Now, let us assume that u and v are two weak solutions of the Problem (14). We double the time variable, see [20], and define for $(t_1, t_2, x) \in (0, T)^2 \times \Omega$:

$$\tilde{u}(t_1, t_2, x) := u(t_1, x), \quad \tilde{v}(t_1, t_2, x) := v(t_2, x).$$

Let $\gamma \in C_0^{\infty}((0, T)^2)$ be nonnegative and η as in the proof of Lemma 3. Further, we define

$$\eta_{\delta}^+(z) := \delta \eta\left(\frac{z}{\delta}\right), \quad \eta_{\delta}^-(z) := \delta \eta\left(-\frac{z}{\delta}\right).$$

Now we use Lemma 2 with $u, \eta_{\delta}^+, w = v(t_2)$, and $\gamma(\cdot, t_2)$, respectively with $v, \eta_{\delta}^-, w = u(t_1)$, and $\gamma(t_1, \cdot)$. By adding up both inequalities and passing to the limit $\delta \rightarrow 0$, we obtain with the same methods as in [18, pages 31-33] the following inequality (here $\text{sign}^+(s) := 1$ for $s \geq 0$ and 0 elsewhere)

$$\begin{aligned} & -\int_0^T \int_0^T \int_{\Omega} (u - v)^+ (\partial_{t_1} + \partial_{t_2}) \gamma dx dt_1 dt_2 \\ & \leq \sum_{i=1}^2 \int_0^T \int_0^T \int_{\Omega_i} \text{sign}^+(u_i - v_i) [f_i(u_i) - f_i(v_i)] \gamma dx dt_1 dt_2. \end{aligned} \quad (19)$$

For nonnegative $\gamma \in C_0^{\infty}(0, T)$ and $\phi \in C_0^{\infty}(\mathbb{R})$ a Dirac-function, we choose in the inequality (19) for $0 < \epsilon \ll 1$

$$\gamma_{\epsilon}(t_1, t_2) := \frac{1}{\epsilon} \phi\left(\frac{t_1 - t_2}{\epsilon}\right) \gamma\left(\frac{t_1 + t_2}{2}\right).$$

In the following, we use the notation $w^{\tau}(t) := w(t - \tau)$. As in [18, page 33] we obtain from (19)

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{\epsilon} \phi\left(\frac{\tau}{\epsilon}\right) \int_0^T \int_{\Omega} - (u - v^{\tau})^+ \partial_t \gamma^{\frac{\tau}{\epsilon}} dx dt d\tau \\ & \leq \sum_{i=1}^2 \int_{-\infty}^{\infty} \frac{1}{\epsilon} \phi\left(\frac{\tau}{\epsilon}\right) \int_0^T \int_{\Omega_i} \text{sign}^+(u_i - v_i)^+ [f_i(u_i) - f_i(v_i)^{\tau}] \gamma^{\frac{\tau}{\epsilon}} dx dt d\tau, \end{aligned}$$

where we used the compact support of $\phi(\frac{\cdot}{\epsilon})$ in $(-\epsilon, \epsilon)$ and $\gamma \in (0, T)$ (here we assumed that ϵ is small enough). By passing to the limit $\epsilon \rightarrow 0$ we obtain (for more details see [18])

$$\int_0^T \int_{\Omega} - (u - v)^+ \partial_t \gamma dx dt \leq \sum_{i=1}^2 \int_0^T \int_{\Omega} \text{sign}^+(u_i - v_i) [f_i(u_i) - f_i(v_i)] \gamma dx dt. \quad (20)$$

We have to show, that the above inequality is also valid for $\gamma \in C_0^\infty([0, T])$. This follows from Lemma 3. In fact, let β_n be a smooth cut-off function with $\beta_n(s) = 0$ for $s \leq 0$ and $\beta_n(s) = 1$ for $s \geq \frac{1}{n}$, and $\beta_n' \geq 0$. By replacing γ in the inequality above with $\beta_n \gamma$ for $\gamma \in C_0^\infty([0, T])$, we obtain

$$\begin{aligned} & - \int_0^T \int_\Omega (u - v)^+ \beta_n \partial_t \gamma \, dx dt - n \int_0^{\frac{1}{n}} \int_\Omega (u - v)^+ \gamma \, dx dt \\ & \leq \sum_{i=1}^2 \int_0^T \int_\Omega \text{sign}^+(u_i - v_i) [f_i(u_i) - f_i(v_i)] \beta_n \gamma \, dx dt. \end{aligned}$$

Due to Lemma 3, the second term on the left-hand side vanishes for $n \rightarrow \infty$ and we obtain the validity of (20) for all $\gamma \in C_0^\infty([0, T])$. Now, we choose $\gamma = \alpha \in C_0^\infty([0, T])$ in (20) and use the Lipschitz continuity of f_i to obtain

$$- \int_0^T \alpha'(t) \int_\Omega (u - v)^+ \, dx dt \leq C \sum_{i=1}^2 \int_0^T \alpha(t) \int_{\Omega_i} (u_i - v_i)^+ \, dx dt.$$

By the same arguments as in the proof of Lemma 3, we obtain the desired result. \square

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